

Global well-posedness for the Schrödinger map problem with small Besov norm

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January 31, 2017

Abstract: In this paper we prove a global result for the Schrödinger map problem with initial data with small Besov norm at critical regularity.

1 Introduction

In this paper we consider the Schrödinger map initial value problem

$$\begin{aligned}\partial_t \phi &= \phi \times \Delta \phi, & \text{on } \mathbf{R} \times \mathbf{R}^d, \\ \phi(0) &= \phi_0,\end{aligned}\tag{1.1}$$

in dimensions $d \geq 3$, where $\phi : \mathbf{R}^d \times \mathbf{R} \rightarrow S^2 \hookrightarrow \mathbf{R}^3$ is a continuous function. The Schrödinger map equation arises in ferromagnetism as the Heisenberg model for the ferromagnetic spin system.

In this paper we are concerned with the issue of global well - posedness of (1.1) for data which is small in a critical space under the scaling.

Observe that for any $\lambda > 0$, if $\phi(t, x)$ solves (1.1), then

$$\phi(\lambda^2 t, \lambda x)\tag{1.2}$$

also solves (1.1). Now for $Q \in S^2$, $\mathbf{Z}_+ = \{0, 1, 2, \dots\}$, and $\sigma \in [0, \infty)$, define the space

$$H_Q^\sigma = \{f : \mathbf{R}^d \rightarrow \mathbf{R}^3, \quad |f(x)| \equiv 1 \quad a.e. \quad f - Q \in H^\sigma\},\tag{1.3}$$

where H^σ is the usual inhomogeneous Sobolev space. This metric has the induced distance

$$d_Q^\sigma(f, g) = \|f - g\|_{H^\sigma(\mathbf{R}^d)}.\tag{1.4}$$

Also define the metric spaces

$$H^\infty = \cap_{\sigma \in \mathbf{Z}_+} H^\sigma, \quad \text{and} \quad H_Q^\infty = \cap_{\sigma \in \mathbf{Z}_+} H_Q^\sigma.\tag{1.5}$$

The scaling law (1.2) preserves the $\dot{H}^{d/2}$ and $\dot{H}_Q^{d/2}$ homogeneous Sobolev norms, where

$$\|f\|_{\dot{H}^\sigma} = \|\mathcal{F}(f)(\xi) \cdot |\xi|^\sigma\|_{L^2}, \quad (1.6)$$

$$\|f\|_{\dot{H}_Q^\sigma} = \|f - Q\|_{\dot{H}^\sigma}, \quad (1.7)$$

and $\mathcal{F} : L^2 \rightarrow L^2$ is the usual Fourier transform

$$\mathcal{F}(f)(\xi) = c_d \int_{\mathbf{R}^d} e^{-ix \cdot \xi} f(x) dx. \quad (1.8)$$

(1.1) is globally well - posed for data which is small in $\dot{H}^{d/2}$.

Theorem 1.1 (Global regularity) *Assume $d \geq 2$ and $Q \in S^2$. Then there exists $\epsilon_0(d) > 0$ such that for any $\phi_0 \in H_Q^\infty$ with $\|\phi_0 - Q\|_{\dot{H}^{d/2}} \leq \epsilon_0(d)$, then there is a unique solution*

$$\phi = S_Q(\phi_0) \in C(\mathbf{R} : H_Q^\infty) \quad (1.9)$$

of the initial value problem (1.1). Moreover,

$$\sup_{t \in \mathbf{R}} \|\phi(t) - Q\|_{\dot{H}^{d/2}} \leq C \|\phi_0 - Q\|_{\dot{H}^{d/2}}, \quad (1.10)$$

and for any $T \in [0, \infty)$ and $\sigma \in \mathbf{Z}_+$,

$$\sup_{t \in [-T, T]} \|\phi(t)\|_{H_Q^\sigma} \leq C(\sigma, T, \|\phi\|_{H_Q^\sigma}). \quad (1.11)$$

This theorem was proved in dimensions $d \geq 4$ in [2], and then for dimensions $d \geq 2$ in [3]. [3] also proved a uniform global bound.

Theorem 1.2 (Uniform bounds and well - posedness) *Assume $d \geq 2$, $Q \in S^2$, and $\sigma_1 \geq \frac{d}{2}$. Then there exists $0 < \epsilon_0(d, \sigma_1) \leq \epsilon_0(d)$ such that if $\phi \in H_Q^\infty$ with $\|\phi - Q\|_{\dot{H}^{d/2}} \leq \epsilon_0(d, \sigma_1)$, then the global solution $\phi(t)$ to (1.1) with initial data ϕ_0 satisfies*

$$\sup_{t \in \mathbf{R}} \|\phi(t) - Q\|_{H^\sigma} \leq C_\sigma \|\phi_0 - Q\|_{H^\sigma}, \quad \frac{d}{2} \leq \sigma \leq \sigma_1. \quad (1.12)$$

In addition, the solution operator admits a continuous extension from

$$B_{\epsilon_0}^\sigma = \{\phi \in \dot{H}_Q^{d/2-1} \cap \dot{H}^\sigma : \|\phi - Q\|_{\dot{H}^{d/2}} \leq \epsilon_0\} \quad (1.13)$$

to $C(\mathbf{R}; \dot{H}^\sigma \cap \dot{H}_Q^{d/2-1})$.

The case when $d = 2$ is particularly interesting, due to the fact that a solution to (1.1) conserves the quantities

$$E_0(t) = \int_{\mathbf{R}^d} |\phi(t) - Q|^2 dx, \quad (1.14)$$

and

$$E_1(t) = \int_{\mathbf{R}^d} \sum_{m=1}^d |\partial_m \phi(t, x)|^2 dx. \quad (1.15)$$

However, in general the proofs in theorems 1.1 and 1.2 are more difficult in lower dimensions. Indeed, [3] utilized the caloric gauge of [18] to analyze dimension $d = 2$, because the Coulomb gauge used in [3] was not strong enough.

In this paper we will extend theorems 1.1 and 1.2 to data which is small in Besov - type norms.

Theorem 1.3 *Suppose $\phi_0 \in \dot{H}^{d/2}$ and suppose that $\psi(x) \in C_0^\infty(\mathbf{R}^d)$ is a radial function supported on $\frac{1}{4} \leq |x| \leq 4$ and $\phi(x) = 1$ on $\frac{1}{2} \leq |x| \leq 2$. Furthermore, suppose that for some $\epsilon_0(d, \|\phi_0\|_{\dot{H}^{d/2}})$,*

$$\sup_j \|\psi(2^j \xi) |\xi|^{d/2} \cdot \mathcal{F}(\phi - Q)(\xi)\|_{L^2(\mathbf{R}^d)} \leq \epsilon_0(d, \|\phi_0\|_{\dot{H}^{d/2}}). \quad (1.16)$$

Then the results of theorems 1.1 and 1.2 hold.

In this paper we follow [1], [2], [3], [10], [11], [12], and [14] and prove a priori bounds on the derivative of the Schrödinger map rather than the Schrödinger map itself. We use the Coulomb gauge, as is typically used in analysis of higher dimensions, see for example [2].

The main new ingredient to this paper is the bilinear estimates obtained from the interaction Morawetz estimates. In [7], applying the bilinear estimates of [13] to the various Schrödinger map gauges gave bilinear estimates for solutions to (1.1) in dimensions $d \geq 2$. These results were used in [8] to improve the results of [15], [16], and [17].

Here we are able to use bilinear arguments in order to prove theorem 1.3. These arguments greatly simplify the function spaces that are needed in the analysis.

2 Bilinear Virial estimates

The work in this paper will heavily rely on bilinear estimates for solutions to the linear Schrödinger equation

$$iu_t + \Delta u = 0. \quad (2.1)$$

Theorem 2.1 (Interaction Morawetz estimate) *If u solves (2.1) then*

$$\| |\nabla|^{\frac{3-d}{2}} |u|^2 \|_{L_{t,x}^2(\mathbf{R} \times \mathbf{R}^d)}^2 \lesssim \|u\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 \|u\|_{L_t^\infty \dot{H}_x^{1/2}(\mathbf{R} \times \mathbf{R}^d)}^2. \quad (2.2)$$

Proof: This theorem was proved in [6] when $d = 3$. [19] subsequently extended the result to dimensions $d \geq 4$. [13] and [5] independently proved theorem 2.1 in dimensions $d = 1, 2$. \square

Of course, [6], [19], [13], and [5] also proved theorem 2.1 for nonlinear Schrödinger equations as well. [13] also made some very important insights into the behavior of linear solutions as well. In particular, [13] observed that one may compute the interaction Morawetz estimate for two different solutions to (2.1), yielding a bilinear estimate.

Theorem 2.2 (Bilinear virial estimate) *Suppose u and v are solutions to (2.1). Then*

$$\begin{aligned} & \int \int_{x_1=y_1} |\partial_1(u(t, x_1, \dots, x_d) \overline{v(t, x_1, y_2, \dots, y_d)})|^2 dx dy dt \\ & \lesssim \|u\|_{L_t^\infty \dot{H}^{1/2}(\mathbf{R} \times \mathbf{R}^d)}^2 \|v\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 + \|v\|_{L_t^\infty \dot{H}^{1/2}(\mathbf{R} \times \mathbf{R}^d)}^2 \|u\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2. \end{aligned} \quad (2.3)$$

In particular, suppose that u and v are solutions to (2.1) and P_j is the usual Littlewood - Paley projection operator. That is, let $\psi(x) = 1$ for $|x| \leq 1$ and $\psi(x) = 0$ for $|x| > 2$ be a smooth, radially symmetric function, decreasing as $r \rightarrow \infty$. Then let

$$\psi_j(x) = \psi(2^{-j}x) - \psi(2^{-j+1}x), \quad (2.4)$$

and let P_j be the Fourier multiplier given by

$$\mathcal{F}(P_j f)(\xi) = [\psi(2^{-j}\xi) - \psi(2^{-j+1}\xi)]\mathcal{F}(f)(\xi). \quad (2.5)$$

Then by elementary Fourier analysis,

$$(P_j f)(x) = 2^{jd} \int \check{\psi}(2^j(x-y))f(y)dy, \quad (2.6)$$

where $\check{\psi}$ is a smooth function that is rapidly decreasing for $|x|$ large, that is, for any integer N ,

$$|\check{\psi}(x)| \lesssim_N (1 + |x|)^{-N}. \quad (2.7)$$

Since the Littlewood - Paley operator is a Fourier multiplier, if u solves (2.1) then $P_j u$ also solves (2.1), so

$$\begin{aligned}
& \int \int_{x_1=y_1} |\partial_1(P_j u(t, x_1, x_2, \dots, x_d) \overline{P_k v(t, x_1, y_2, \dots, y_d)})|^2 dx dy dt \\
& \lesssim \|P_j u\|_{L_t^\infty \dot{H}^{1/2}(\mathbf{R} \times \mathbf{R}^d)}^2 \|P_k v\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 + \|P_k v\|_{L_t^\infty \dot{H}^{1/2}(\mathbf{R} \times \mathbf{R}^d)}^2 \|P_j u\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2.
\end{aligned} \tag{2.8}$$

But then by (2.6), (2.7), (2.8), and Hölder's inequality, if $k \leq j$,

$$\|\partial_1(P_j u \overline{P_k v})\|_{L_{t,x}^2(\mathbf{R} \times \mathbf{R}^d)}^2 \lesssim 2^{(d-1)k} 2^j \|P_j u\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 \|P_k v\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2. \tag{2.9}$$

Now since there is absolutely nothing special about the direction $e_1 \in S^{d-1}$, averaging over all directions $\omega \in S^1$ gives

$$\|\nabla(P_j u \overline{P_k v})\|_{L_{t,x}^2(\mathbf{R} \times \mathbf{R}^d)}^2 \lesssim 2^{(d-1)k} 2^j \|P_j u\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 \|P_k v\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2. \tag{2.10}$$

Thus, if $j \geq k + 10$, then by the Fourier support of $P_j u \cdot \overline{P_k v}$, and Bernstein's inequality,

$$\|(P_j u \overline{P_k v})\|_{L_{t,x}^2(\mathbf{R} \times \mathbf{R}^d)}^2 \lesssim 2^{(d-1)k} 2^{-j} \|P_j u\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 \|P_k v\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2. \tag{2.11}$$

Also, by theorem 1.1, when $d \geq 3$, and $j - 10 \leq k \leq j$,

$$\|(P_j u \overline{P_k v})\|_{L_{t,x}^2(\mathbf{R} \times \mathbf{R}^d)}^2 \lesssim 2^{(d-1)k} 2^{-j} \|P_j u\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 \|P_k v\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2. \tag{2.12}$$

Remark: This will not work in dimensions $d = 1, 2$ because we can merely say that $P_j u \cdot \overline{P_k v}$ is supported on $|\xi| \lesssim 2^k$ when $j - 10 \leq k \leq j$, so when the power of $|\nabla|^{\frac{3-d}{2}}$ is positive, then (2.12) does not follow directly from theorem 1.1 and the above analysis. It is for this technical reason that this paper only addresses dimensions $d \geq 3$.

It is appropriate to point out that (2.11) has been proven to be true in [4] using Fourier analysis. In fact, [4] and subsequent Fourier analytic proofs in higher dimensions all predate the bilinear virial arguments of [13]. However, in this case, it is quite useful to examine the bilinear estimates through the lens of the virial identity. In [13] the interaction Morawetz for theorem 2.2 is

$$\begin{aligned}
M(t) &= \int |v(t, y)|^2 \frac{(x-y)_1}{|(x-y)_1|} \operatorname{Im}[\bar{u} \partial_1 u](t, x) dx dy \\
&+ \int |u(t, y)|^2 \frac{(x-y)_1}{|(x-y)_1|} \operatorname{Im}[\bar{v} \partial_1 v](t, x) dx dy.
\end{aligned} \tag{2.13}$$

Averaging over all directions in S^{d-1} , suppose u is the solution to

$$iu_t + \Delta u = \mathcal{N}_1, \quad (2.14)$$

and v is the solution to

$$iv_t + \Delta v = \mathcal{N}_2. \quad (2.15)$$

For any \mathbf{R}^d , for some c_d , for any $y \in \mathbf{R}^d$,

$$\int_{S^{d-1}} \frac{x \cdot \omega}{|x \cdot \omega|} (y \cdot \omega) d\omega = c_d \frac{x \cdot y}{|x|}. \quad (2.16)$$

Without loss of generality suppose $x = |x|e_1$. Then,

$$\frac{1}{2} \frac{x \cdot \omega}{|x \cdot \omega|} y \cdot \omega + \frac{1}{2} \frac{x \cdot (-\omega)}{|x \cdot (-\omega)|} (y \cdot (-\omega)) = \frac{x_1 \omega_1^2}{|x_1| |\omega_1|} y_1, \quad (2.17)$$

which implies (2.16). Then by theorem 2.2 and averaging over $\omega \in S^{d-1}$, taking $\partial_\omega f = \omega \cdot \nabla f$, and $x_\omega = x \cdot \omega$,

$$\begin{aligned} & \int_{S^{d-1}} \int_{x_\omega=y_\omega} |\partial_\omega (u(t, y) \overline{v(t, x)})|^2 dx dy dt d\omega \\ & \lesssim \|u\|_{L_t^\infty \dot{H}^{1/2}(\mathbf{R} \times \mathbf{R}^d)}^2 \|v\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 + \|v\|_{L_t^\infty \dot{H}^{1/2}(\mathbf{R} \times \mathbf{R}^d)}^2 \|u\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 \\ & \quad + \int |v(t, y)|^2 \frac{(x-y)}{|(x-y)|} \cdot \text{Im}[\overline{u} \nabla \mathcal{N}_1](t, x) dx dy dt \\ & \quad + \int |v(t, y)|^2 \frac{(x-y)}{|(x-y)|} \cdot \text{Im}[\overline{\mathcal{N}_1} \nabla u](t, x) dx dy dt \\ & \quad + \int |u(t, y)|^2 \frac{(x-y)}{|(x-y)|} \cdot \text{Im}[\overline{v} \nabla \mathcal{N}_2](t, x) dx dy dt \\ & \quad + \int |u(t, y)|^2 \frac{(x-y)}{|(x-y)|} \cdot \text{Im}[\overline{\mathcal{N}_2} \nabla v](t, x) dx dy dt \\ & \quad + \int \text{Re}[\overline{v} \mathcal{N}_2](t, y) \frac{(x-y)}{|(x-y)|} \cdot \text{Im}[\overline{u} \nabla u](t, x) dx dy dt \\ & \quad + \int \text{Re}[\overline{u} \mathcal{N}_1](t, y) \frac{(x-y)}{|(x-y)|} \cdot \text{Im}[\overline{v} \nabla v](t, x) dx dy dt. \end{aligned} \quad (2.18)$$

Then by Hölder's inequality and averaging over $\omega \in S^{d-1}$, for any $R > 0$,

$$\begin{aligned}
& \frac{1}{R} \int \int_{h \in \mathbf{R}^d: |h| \leq R} \int_{\mathbf{R}^d} |\nabla(u(t, x) \overline{v(t, x+h)})|^2 dx dh dt \\
& \lesssim \|u\|_{L_t^\infty \dot{H}^{1/2}(\mathbf{R} \times \mathbf{R}^d)}^2 \|v\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 + \|v\|_{L_t^\infty \dot{H}^{1/2}(\mathbf{R} \times \mathbf{R}^d)}^2 \|u\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 \\
& \quad + \int |v(t, y)|^2 \frac{(x-y)}{|(x-y)|} \cdot \text{Im}[\overline{u} \nabla \mathcal{N}_1](t, x) dx dy dt \\
& \quad + \int |v(t, y)|^2 \frac{(x-y)}{|(x-y)|} \cdot \text{Im}[\overline{\mathcal{N}_1} \nabla u](t, x) dx dy dt \\
& \quad + \int |u(t, y)|^2 \frac{(x-y)}{|(x-y)|} \cdot \text{Im}[\overline{v} \nabla \mathcal{N}_2](t, x) dx dy dt \\
& \quad + \int |u(t, y)|^2 \frac{(x-y)}{|(x-y)|} \cdot \text{Im}[\overline{\mathcal{N}_2} \nabla v](t, x) dx dy dt \\
& \quad + \int \text{Re}[\overline{v} \mathcal{N}_2](t, y) \frac{(x-y)}{|(x-y)|} \cdot \text{Im}[\overline{u} \nabla u](t, x) dx dy dt \\
& \quad + \int \text{Re}[\overline{u} \mathcal{N}_1](t, y) \frac{(x-y)}{|(x-y)|} \cdot \text{Im}[\overline{v} \nabla v](t, x) dx dy dt.
\end{aligned} \tag{2.19}$$

Therefore, for $k \leq j$,

$$\begin{aligned}
& \|(P_j u \overline{P_k v})\|_{L_{t,x}^2(\mathbf{R} \times \mathbf{R}^d)}^2 \lesssim 2^{(d-1)k} 2^{-j} \|P_j u\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 \|P_k v\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}^2 \\
& \quad + 2^{(d-1)k} 2^{-2j} \int |P_k v(t, y)|^2 \frac{(x-y)}{|(x-y)|} \cdot \text{Re}[\overline{P_j u} \nabla P_j \mathcal{N}_1](t, x) dx dy dt \\
& \quad + 2^{(d-1)k} 2^{-2j} \int |P_k v(t, y)|^2 \frac{(x-y)}{|(x-y)|} \cdot \text{Re}[\overline{P_j \mathcal{N}_1} \nabla P_j u](t, x) dx dy dt \\
& \quad + 2^{(d-1)k} 2^{-2j} \int |P_j u(t, y)|^2 \frac{(x-y)}{|(x-y)|} \cdot \text{Re}[\overline{P_k v} \nabla P_k \mathcal{N}_2](t, x) dx dy dt \\
& \quad + 2^{(d-1)k} 2^{-2j} \int |P_j u(t, y)|^2 \frac{(x-y)}{|(x-y)|} \cdot \text{Re}[\overline{P_k \mathcal{N}_2} \nabla P_k v](t, x) dx dy dt \\
& \quad + 2^{(d-1)k} 2^{-2j} \int \text{Re}[\overline{P_k v} P_k \mathcal{N}_2](t, y) \frac{(x-y)}{|(x-y)|} \cdot \text{Re}[P_j \overline{u} \nabla P_j u](t, x) dx dy dt \\
& \quad + 2^{(d-1)k} 2^{-2j} \int \text{Re}[\overline{P_j u} P_j \mathcal{N}_1](t, y) \frac{(x-y)}{|(x-y)|} \cdot \text{Re}[P_k \overline{v} \nabla P_k v](t, x) dx dy dt.
\end{aligned} \tag{2.20}$$

This is a useful estimate since it opens up the possibility of integrating by parts, and moving a derivative to a more advantageous position. This opportunity will prove to be quite useful, since we will follow the analysis in [2]. That is, if ϕ be a solution to (1.1), let

$$\psi_x = \nabla \phi = (\psi_1, \dots, \psi_d) = (\partial_1 \phi, \dots, \partial_d \phi). \tag{2.21}$$

Then, choosing to work in the Coulomb gauge, for $m = 1, \dots, d$,

$$(i\partial_t + \Delta_x)\psi_m = -2i \sum_{l=1}^d A_l \cdot \partial_l \psi_m + (A_0 + \sum_{l=1}^d A_l^2)\psi_m - i \sum_{l=1}^d \text{Im}(\psi_m \bar{\psi}_l) \psi_l, \quad (2.22)$$

where

$$A_l = - \sum_{k=1}^d \frac{\partial_k}{\Delta} \text{Im}(\psi_l \bar{\psi}_k), \quad (2.23)$$

and

$$A_0 = \sum_{l,m=1}^d \frac{\partial_l \partial_m}{\Delta} (\text{Re}(\bar{\psi}_l \psi_m)) - \frac{1}{2} \left(\sum_{m=1}^d \psi_m \bar{\psi}_m \right). \quad (2.24)$$

3 Growth of Besov norms

Now we utilize frequency envelopes to control the growth of the Besov norm. The frequency envelopes were introduced in [18] for the study of wave maps. Frequency envelopes majorize the size of a Littlewood - Paley projection of a function, while at the same time smoothing out the differences in size at different frequency levels. Let $\delta > 0$ be a small constant, say $\delta = \frac{1}{4}$, and let

$$\alpha_j(0) = \sup_k 2^{-\delta|j-k|} 2^{k(\frac{d-2}{2})} \|P_k \psi_x(0)\|_{L^2(\mathbf{R}^d)}, \quad (3.1)$$

$$\alpha_j = \sup_k 2^{-\delta|j-k|} 2^{k(\frac{d-2}{2})} \|P_k \psi_x\|_{L_t^\infty L_x^2(\mathbf{R} \times \mathbf{R}^d)}, \quad (3.2)$$

$$\beta_j = \sup_k 2^{-(\delta/2)|j-k|} 2^{k(\frac{d-2}{4})} \|P_k \psi_x\|_{L_{t,x}^4(\mathbf{R} \times \mathbf{R}^d)}, \quad (3.3)$$

and

$$\begin{aligned} \gamma_j &= \sup_k 2^{-\delta|j-k|} 2^{k(\frac{d-2}{2})} \left(\sup_{l \leq k} 2^{\frac{1}{2}(k-l)} \|(P_k \psi_x)(P_l \psi_x)\|_{L_{t,x}^2(\mathbf{R} \times \mathbf{R}^d)} \right) \\ &+ \sup_k 2^{-\delta|j-k|} 2^{k(\frac{d-2}{2})} \left(\sup_{l \leq k} \sup_{R>0} \frac{1}{R} 2^k 2^{(d-2)l} \int_{h \in \mathbf{R}^d, |h| \leq R} |P_k \psi_x(t, x)|^2 |P_l \psi_x(t, x+h)|^2 dx dy dt \right)^{1/2}. \end{aligned} \quad (3.4)$$

Remark: From the definition,

$$\beta_j^2 \lesssim \gamma_j. \quad (3.5)$$

Also observe that for any j ,

$$\|P_j \psi_x(0)\|_{\dot{H}^{\frac{d-2}{2}}} \leq \alpha_j(0), \quad (3.6)$$

and by Young's inequality,

$$\|\psi_x(0)\|_{\dot{H}^{\frac{d-2}{2}}}^2 \sim \sum_j \alpha_j(0)^2. \quad (3.7)$$

Now if $\langle f, g \rangle$ is the inner product

$$Re \int f(x) \overline{g(x)} dx, \quad (3.8)$$

then

$$\|P_j \psi_x\|_{L_x^2(\mathbf{R}^d)}^2 = \langle P_j \psi_x, P_j \psi_x \rangle, \quad (3.9)$$

so by (2.22),

$$\begin{aligned} \frac{d}{dt} \langle P_j \psi_x, P_j \psi_x \rangle &= -4 \langle P_j \left(\sum_{l=1}^d A_l \partial_l \psi_x \right), P_j \psi_x \rangle \\ &\quad - 2 \langle i P_j ((A_0 + \sum_{l=1}^d A_l^2) \psi_x), P_j \psi_x \rangle - 2 \langle P_j \left(\sum_{l=1}^d Im(\psi_x \bar{\psi}_l) \psi_l \right), P_j \psi_x \rangle. \end{aligned} \quad (3.10)$$

Now by Fourier support arguments,

$$\begin{aligned} &\sup_k 2^{-2\delta|j-k|} 2^{k(d-2)} \int_{\mathbf{R}} |\langle P_k \left(\sum_{l=1}^d Im(\psi_x \bar{\psi}_l) \psi_l \right), P_k \psi_x \rangle| dt \\ &\lesssim \sup_k 2^{-2\delta|j-k|} 2^{k(d-2)} \|P_k \psi_x\|_{L_{t,x}^4} \|P_{\geq k-10} \psi_x\|_{L_{t,x}^4}^3 \\ &+ \sup_k 2^{-2\delta|j-k|} 2^{k(d-2)} \|(P_k \psi_x)(P_{\leq k-10} \psi_x)\|_{L_{t,x}^2} \|(P_{k-10 \leq \cdot \leq k+10} \psi_x)(P_{\leq k-10} \psi_x)\|_{L_{t,x}^2} \\ &\lesssim \beta_j^4 + \gamma_j^2 \lesssim \gamma_j^2. \end{aligned} \quad (3.11)$$

By a similar calculation,

$$\sup_k 2^{-2\delta|j-k|} 2^{k(d-2)} \int_{\mathbf{R}} |\langle i P_k \left(-\frac{1}{2} \left(\sum_{m=1}^d \psi_m \bar{\psi}_m \right) \psi_x \right), P_k \psi_x \rangle| dt \lesssim \beta_j^4 + \gamma_j^2 \lesssim \gamma_j^2. \quad (3.12)$$

Now split

$$\begin{aligned} &\frac{\partial_l \partial_m}{\Delta} Re(\bar{\psi}_l \psi_m) = \frac{\partial_l \partial_m}{\Delta} Re((P_{\geq k-20} \bar{\psi}_l)(P_{\geq k-20} \psi_m)) \\ &+ 2 \frac{\partial_l \partial_m}{\Delta} Re((P_{\geq k-20} \bar{\psi}_l)(P_{\leq k-20} \psi_m)) + \frac{\partial_l \partial_m}{\Delta} Re((P_{\leq k-20} \bar{\psi}_l)(P_{\leq k-20} \psi_m)). \end{aligned} \quad (3.13)$$

Because $\frac{\partial_l \partial_m}{\Delta}$ is a bounded Fourier multiplier,

$$\begin{aligned}
& \sup_k 2^{-2\delta|j-k|} 2^{k(d-2)} \int |\langle \frac{\partial_l \partial_m}{\Delta} (Re(P_{\geq k-20} \bar{\psi}_l)(P_{\geq k-20} \psi_m)) \psi_x, P_k \psi_x \rangle| dt \\
& \lesssim \sup_k 2^{-2\delta|j-k|} 2^{k(d-2)} \|P_{\geq k-20} \psi_x\|_{L_{t,x}^4}^2 \|P_k \psi_x\|_{L_{t,x}^4} \|P_{\geq k-10} \psi_x\|_{L_{t,x}^4} \\
& + \sup_k 2^{-2\delta|j-k|} 2^{k(d-2)} \|P_{\geq k-20} \psi_x\|_{L_{t,x}^4}^2 \|(P_k \psi_x)(P_{\leq k-10} \psi_x)\|_{L_{t,x}^2} \lesssim \beta_j^4 + \gamma_j^2.
\end{aligned} \tag{3.14}$$

Meanwhile, because $(P_{\leq k-20} \bar{\psi}_l)(P_{\leq k-20} \psi_m)$ is supported on $|\xi| \leq 2^{k-15}$, by (2.7) we compute

$$\begin{aligned}
& \sup_k 2^{-2\delta|j-k|} 2^{(d-2)k} \int \sum_{l \leq k-15} \int_{\mathbf{R}^d} 2^{ld} |\check{\psi}(2^l(x-y))| |P_{l \leq \cdot \leq k-20} \psi_x(t, y)|^2 \\
& \quad \times |P_k \psi_x(t, x)| |P_{k-10 \leq \cdot \leq k+10} \psi_x(t, x)| dx dy dt \tag{3.15} \\
& \lesssim \gamma_j^2 \sum_{l \leq k-20} 2^{(d-1)l} \sum_{l \leq m \leq k-20} 2^{-(d-2)m} 2^{-(d-2)k} 2^{-k} \lesssim \gamma_j^2.
\end{aligned}$$

Also by (2.7),

$$\begin{aligned}
& \sup_k 2^{-2\delta|j-k|} 2^{(d-2)k} \int \sum_{l \leq k-15} \int_{\mathbf{R}^d} 2^{ld} |\check{\psi}(2^l(x-y))| |P_{\leq l} \psi_x(t, y)|^2 \\
& \quad \times |P_k \psi_x(t, x)| |P_{k-10 \leq \cdot \leq k+10} \psi_x(t, x)| dx dy dt \\
& \lesssim \sup_k 2^{-2\delta|j-k|} 2^{(d-2)k} \sum_{l \leq k-15} \|(P_k \psi_x)(P_{\leq l} \psi_x)\|_{L_{t,x}^2} \|(P_{k-10 \leq \cdot \leq k+10} \psi_x)(P_{\leq l} \psi_x)\|_{L_{t,x}^2} \lesssim \gamma_j^2.
\end{aligned} \tag{3.16}$$

Therefore, combining (3.12) - (3.16) with an interpolation argument proves

$$\sup_k 2^{-2\delta|j-k|} 2^{k(d-2)} |\langle i P_k ((A_0) \psi_x), P_k \psi_x \rangle| dt \lesssim \gamma_j^2. \tag{3.17}$$

Next, integrating by parts, since $\sum_{l=1}^d \partial_l A_l = 0$

$$-4 \langle \sum_{l=1}^d A_l \cdot \partial_l (P_k \psi_x), P_k \psi_x \rangle = 0. \tag{3.18}$$

To estimate

$$-4 \langle \sum_{l=1}^d [P_k, A_l] \partial_l \psi_x, P_k \psi_x \rangle, \tag{3.19}$$

observe that if m is a Fourier multiplier satisfying $|\nabla m(\xi)| \lesssim \frac{1}{|\xi|}$ and $|m(\xi)| \lesssim 1$, then by the fundamental theorem of calculus, if $|\eta| << |\xi|$,

$$|\xi| |m(\xi + \eta) \hat{f}(\eta) \hat{g}(\xi) - m(\xi) \hat{f}(\eta) \hat{g}(\xi)| \lesssim |\eta| |\hat{f}(\eta)| |\hat{g}(\xi)|. \tag{3.20}$$

If $|\eta| \gtrsim |\xi|$,

$$|\xi| |m(\xi + \eta) \hat{f}(\eta) \hat{g}(\xi)| + |\xi| |m(\xi) \hat{f}(\eta) \hat{g}(\xi)| \lesssim |\eta| |\hat{f}(\eta)| |\hat{g}(\xi)|. \quad (3.21)$$

Therefore, by (2.23)

$$2^{-2\delta|j-k|} 2^{k(d-2)} \int |\langle [P_k, A_t] \partial_t \psi_x, P_k \psi_x \rangle| dt \quad (3.22)$$

may be estimated in a manner identical to the argument leading up to (3.17).

It finally remains to estimate

$$\sup_k 2^{-2\delta|j-k|} 2^{(d-2)k} \int |\langle i P_k (\sum_{l=1}^d A_l^2 \psi_x), P_k \psi_x \rangle| dt. \quad (3.23)$$

By the Sobolev embedding theorem and Bernstein's inequality,

$$\begin{aligned} \|\psi_x\|_{L_x^d(\mathbf{R}^d)}^d &\lesssim \sum_{j_1 \leq \dots \leq j_d} \|P_{j_1}\|_{L_x^\infty} \cdots \|P_{j_{d-2}}\|_{L_x^\infty} \|P_{j_{d-1}} \psi_x\|_{L_x^2} \|P_{j_d} \psi_x\|_{L_x^2} \\ &\lesssim \sum_{j_1 \leq \dots \leq j_d} \|P_{j_1} \psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}}^{\frac{d-2}{2}} \cdots \|P_{j_d} \psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}}^{\frac{d-2}{2}} 2^{j_1} \cdots 2^{j_{d-2}} 2^{-j_{d-1}(\frac{d-2}{2})} 2^{-j_d(\frac{d-2}{2})} \\ &\lesssim \|\psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}}^2 \left(\sup_j \|P_j \psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}} \right)^{d-2}. \end{aligned} \quad (3.24)$$

Next, by the Sobolev embedding theorem and (3.24),

$$\|P_k(-\sum_{k=1}^d \frac{\partial_k}{\Delta} \text{Im}(\psi_l \bar{\psi}_k))\|_{L_x^\infty} \lesssim 2^k \|\psi_x\|_{L_x^d}^2 \lesssim 2^k \|\psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}}^{\frac{4}{d}} \left(\sup_j \|P_j \psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}} \right)^{\frac{2(d-2)}{d}}. \quad (3.25)$$

Therefore, by Fourier support properties,

$$\begin{aligned} &\sum_{n \leq m \leq k-10} \| |P_k(P_m A_x \cdot P_n A_x \cdot \psi_x)| |P_k \psi_x| \|_{L_{t,x}^1} \\ &\lesssim \|\psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}}^{\frac{4}{d}} \left(\sup_j \|P_j \psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}} \right)^{\frac{2(d-2)}{d}} \sum_m \| |P_k(2^m P_m A_x \cdot \psi_x)| |P_k \psi_x| \|_{L_{t,x}^1}. \end{aligned} \quad (3.26)$$

Now since the Fourier multiplier $2^m \frac{\partial_k}{\Delta} P_m$ is uniformly bounded, by an argument similar to (3.13) - (3.17),

$$\sup_k 2^{-2\delta|j-k|} 2^{(d-2)k} (3.26) \lesssim \|\psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}}^{\frac{4}{d}} \left(\sup_j \|P_j \psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}} \right)^{\frac{2(d-2)}{d}} (\beta_j^4 + \gamma_j^2) \lesssim \gamma_j^2. \quad (3.27)$$

On the other hand, if $m \geq k - 10$, then by Fourier support arguments and the Sobolev embedding theorem,

$$\|P_n A_x\|_{L_{t,x}^4} \lesssim \|P_{\geq k-15} \psi_x\|_{L_{t,x}^4} \|\psi_x\|_{L_x^d}, \quad (3.28)$$

and therefore,

$$\begin{aligned} & \sup_k 2^{-2\delta|j-k|} 2^{(d-2)k} \int |\langle i P_k (\sum_{l=1}^d A_l^2 \psi_x), P_k \psi_x \rangle| dt \\ & \lesssim \|\psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}}^{\frac{4}{d}} (\sup_j \|P_j \psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}})^{\frac{2(d-2)}{d}} (\beta_j^4 + \gamma_j^2). \end{aligned} \quad (3.29)$$

In conclusion, we have proved

Theorem 3.1 *The frequency envelope α has the bounds*

$$\begin{aligned} \alpha_j & \lesssim \alpha_j(0) + \gamma_j (1 + \|\psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}}^{\frac{2}{d}} (\sup_j \|P_j \psi_x\|_{\dot{H}_x^{\frac{d-2}{2}}})^{\frac{d-2}{d}}) \\ & \lesssim \alpha_j(0) + \gamma_j (1 + (\sum_j \alpha_j^2)^{2/d} (\sup_j \alpha_j^2)^{\frac{d-2}{d}}). \end{aligned} \quad (3.30)$$

4 Bilinear estimates

Now we use the interaction Morawetz estimates to prove some bilinear estimates.

Proposition 4.1 *For any j ,*

$$\begin{aligned} \gamma_j^2 & \lesssim (\sup_k \alpha_k)^2 \alpha_j^2 + \gamma_j^2 (\sup_k \alpha_k)^2 (1 + (\sum_k \alpha_k^2)^{2/d} (\sup_k \alpha_k)^{\frac{2(d-2)}{d}}) \\ & \quad + \alpha_j^2 (\sup_k \gamma_k)^2 (1 + (\sum_k \alpha_k^2)^{2/d} (\sup_k \alpha_k)^{\frac{2(d-2)}{d}}). \end{aligned} \quad (4.1)$$

Proof: Take (2.20) and set $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}$, $u = v = \psi_x$, where

$$\mathcal{N} = -i \sum_{l=1}^d \text{Im}(\psi_x \bar{\psi}_l) \psi_l + A_0 \psi_x + \sum_{l=1}^d A_l^2 \psi_x - 2i \sum_{l=1}^d A_l \cdot \partial_l \psi_x = \mathcal{N}^{(1)} + \mathcal{N}^{(2)} + \mathcal{N}^{(3)} + \mathcal{N}^{(4)}. \quad (4.2)$$

We will call the first line on the right hand side of (2.20) the main term, and the other lines the error. Then we label the contribution to the error from $\mathcal{N}^{(1)}$, $\mathcal{E}^{(1)}$, and so on. The main term may be easily estimated.

$$2^{(d-2)l} 2^{-l} \|P_j \psi_x\|_{L_t^\infty L_x^2}^2 \sup_{k \leq l} (2^{l-k} 2^{(d-1)k} \|P_k \psi_x\|_{L_t^\infty L_x^2}^2) \lesssim \alpha_j^2 (\sup_k \alpha_k)^2. \quad (4.3)$$

Computing $\mathcal{E}^{(1)}$, (3.11) implies

$$\begin{aligned} 2^{l-k}2^{(d-2)l}\mathcal{E}^{(1)} &\lesssim 2^{(d-2)l}2^{(d-2)k}|||P_l\mathcal{N}^{(1)}|||P_l\psi_x|||_{L_{t,x}^1}||P_k\psi_x||_{L_t^\infty L_x^2}^2 \\ &\quad + 2^{(d-2)k}2^{(d-2)l}2^{k-l}|||P_k\mathcal{N}^{(1)}|||P_k\psi_x|||_{L_{t,x}^1}||P_l\psi_x||_{L_t^\infty L_x^2}^2 \\ &\lesssim 2^{2\delta|j-l|}[\gamma_j^2(\sup_k \alpha_k)^2 + \alpha_j^2(\sup_k \gamma_k)^2]. \end{aligned} \quad (4.4)$$

Similarly, by (3.12) - (3.18),

$$2^{l-k}2^{(d-2)l}\mathcal{E}^{(2)} \lesssim 2^{2\delta|j-l|}[\gamma_j^2(\sup_k \alpha_k)^2 + \alpha_j^2(\sup_k \gamma_k)^2], \quad (4.5)$$

and by (3.23) - (3.29),

$$2^{l-k}2^{(d-2)l}\mathcal{E}^{(3)} \lesssim 2^{2\delta|j-l|}[\gamma_j^2(\sup_k \alpha_k)^2 + \alpha_j^2(\sup_k \gamma_k)^2](1 + \|\psi_x\|_{L_t^\infty L_x^d}^2). \quad (4.6)$$

To estimate the integral of $\mathcal{E}^{(4)}$, split

$$-4P_k(A_x \cdot \nabla_x \psi_x) = -4A_x \cdot \nabla_x(P_k\psi_x) - 4[P_k, A_x] \cdot \nabla_x \psi_x = \mathcal{N}_1^{(4)} + \mathcal{N}_2^{(4)}, \quad (4.7)$$

and split $\mathcal{E}^{(4)} = \mathcal{E}_1^{(4)} + \mathcal{E}_2^{(4)}$. Then by (3.20) - (3.22),

$$2^{l-k}2^{(d-2)l}\mathcal{E}_2^{(4)} \lesssim 2^{2\delta|j-l|}[\gamma_j^2(\sup_k \alpha_k)^2 + \alpha_j^2(\sup_k \gamma_k)^2]. \quad (4.8)$$

To estimate the contribution of $\mathcal{E}_1^{(4)}$ we again integrate by parts. First,

$$-4Re[\overline{P_k\psi_x}(A_x \cdot \nabla_x(P_k\psi_x))] = -2Re[A_x \cdot \nabla_x(|P_k\psi_x|^2)]. \quad (4.9)$$

Then integrating by parts,

$$2^{-2l}2^{(d-1)k} \int Re[\overline{P_k\psi_x}(A_x \cdot \nabla_x(P_k\psi_x))](t, y) \frac{(x-y)}{|(x-y)|} \cdot Re[P_l\bar{\psi}_x \nabla P_l\psi_x](t, x) dx dy dt \quad (4.10)$$

$$\lesssim 2^{-2l}2^{(d-1)k} \int |P_k\psi_x(t, y)|^2 |A_x(t, y)| \frac{1}{|x-y|} |\nabla P_l\psi_x(t, x)| |P_l\psi_x(t, x)| dx dy dt. \quad (4.11)$$

Now recall the definition of A_x in (2.23). Then we can split

$$\begin{aligned}
A_x &= \sum_{m=1}^d \frac{\partial_m}{\Delta} \text{Im}(\psi_x \bar{\psi}_m) = \sum_{m=1}^d \frac{\partial_m}{\Delta} \text{Im}((P_{\leq k-10} \psi_x)(P_{\leq k-10} \bar{\psi}_m)) \\
&+ 2 \sum_{m=1}^d \frac{\partial_m}{\Delta} \text{Im}((P_{\leq k-10} \psi_x)(P_{\geq k-10} \bar{\psi}_m)) + \sum_{m=1}^d \frac{\partial_m}{\Delta} \text{Im}((P_{\geq k-10} \psi_x)(P_{\geq k-10} \bar{\psi}_m)) \\
&= A_x^{(1)} + A_x^{(2)} + A_x^{(3)}. \tag{4.12}
\end{aligned}$$

Now by definition of γ_k and β_k , if $\delta < \frac{1}{4}$, then by the Sobolev embedding theorem,

$$\|A_x^{(2)}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} + \|A_x^{(3)}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \lesssim \|P_{\leq k-10} \psi_x\|_{L_{t,x}^2} \|P_{\geq k-10} \psi_x\|_{L_{t,x}^4} + \|P_{\geq k-10} \psi_x\|_{L_{t,x}^4} \lesssim \gamma_k + \beta_k^2. \tag{4.13}$$

Therefore,

$$\begin{aligned}
&2^{-2l} \sup_k (2^{l-k} 2^{(d-1)k}) \int |P_k \psi_x(t, y)|^2 |A_x^{(2)}(t, y)| \frac{1}{|x-y|} |\nabla P_l \psi_x(t, x)| |P_l \psi_x(t, x)| dx dy dt \\
&+ 2^{-2l} \sup_k (2^{l-k} 2^{(d-1)k}) \int |P_k \psi_x(t, y)|^2 |A_x^{(3)}(t, y)| \frac{1}{|x-y|} |\nabla P_l \psi_x(t, x)| |P_l \psi_x(t, x)| dx dy dt \\
&\lesssim 2^{2\delta|j-l|} 2^{-j(d-2)} (\sup_k \beta_k)^2 (\sup_k \gamma_k + (\sup_k \beta_k)^2) \alpha_l^2 \lesssim 2^{2\delta|j-l|} 2^{-l(d-2)} ((\sup_k \gamma_k)^2 + (\sup_k \beta_k)^4) \alpha_j^2. \tag{4.14}
\end{aligned}$$

Now by Fourier support properties,

$$\sum_{m=1}^d \frac{\partial_m}{\Delta} \text{Im}((P_{\leq k-10} \psi_x)(P_{\leq k-10} \bar{\psi}_m)) \tag{4.15}$$

is supported on $|\xi| \leq 2^{k-5}$, so

$$A_x^{(1)} = \sum_{m \leq k-5} P_m A_x^{(1)}. \tag{4.16}$$

The Fourier multiplier $P_m(\frac{\partial_k}{\Delta})$ has size $\sim 2^{-m}$, so by (3.15) and (3.16),

$$\begin{aligned}
&2^{-2l} \sup_k (2^{l-k} 2^{(d-1)k}) \sum_{m \leq k-5} \int_{|x-y| \geq 2^{-k} 2^{\frac{1}{d}(k-m)}} |P_k \psi_x(t, y)|^2 |P_m A_x^{(1)}(t, y)| \\
&\times \frac{1}{|x-y|} |\nabla P_l \psi_x(t, x)| |P_l \psi_x(t, x)| dx dy dt \lesssim 2^{-l(d-2)} 2^{2\delta|j-l|} \alpha_j^2 (\sup_k \gamma_k)^2. \tag{4.17}
\end{aligned}$$

Also, by the Sobolev embedding theorem and Bernstein's inequality,

$$\|P_m A_x^{(1)}\|_{L_x^\infty} \lesssim 2^{-m} \|P_{\leq m} \psi_x\|_{L_x^\infty}^2 + 2^{(d-1)m} \|P_{\geq m} \psi_x\|_{L_x^2}^2 \lesssim 2^m (\sup_k \alpha_k)^2. \tag{4.18}$$

Therefore, by Hölder's inequality,

$$\begin{aligned}
& 2^{-2l} \sup_k (2^{l-k} 2^{(d-1)k}) \sum_{m \leq k-5} \int_{|x-y| \leq 2^{-k} 2^{\frac{1}{d}(k-m)}} |P_k \psi_x(t, y)|^2 |P_m A_x^{(1)}(t, y)| \\
& \quad \times \frac{1}{|x-y|} |\nabla P_l \psi_x(t, x)| |P_l \psi_x(t, x)| dx dy dt \\
& \lesssim \sup_k \|(P_l \psi_x)(P_k \psi_x)\|_{L_{t,x}^2}^2 (\sup_k \alpha_k^2) \lesssim 2^{2\delta|j-l|} 2^{-l(d-2)} \gamma_j^2 (\sup_k \alpha_k)^2.
\end{aligned} \tag{4.19}$$

Also,

$$-4 \operatorname{Re}[\overline{P_l \psi_x}(A_x \cdot \nabla_x(P_l \psi_x))] = -2 \operatorname{Re}[A_x \cdot \nabla_x(|P_l \psi_x|^2)], \tag{4.20}$$

so following the analysis in (4.9) - (4.19) with l and k swapped proves

$$\begin{aligned}
& 2^{-2l} \sup_k 2^{(d-1)k} \int \operatorname{Re}[\overline{P_l \psi_x}(A_x \cdot \nabla_x(P_l \psi_x))](t, y) \frac{(x-y)}{|(x-y)|} \cdot \operatorname{Re}[P_k \bar{\psi}_x \nabla P_k \psi_x](t, x) dx dy dt \\
& \lesssim 2^{-l(d-2)} 2^{2\delta|j-l|} \gamma_j^2 (\sup_k \alpha_k)^2.
\end{aligned} \tag{4.21}$$

This finally proves proposition 4.1. \square

Proof of theorem 1.3: Theorem 1.3 may be proved by a bootstrap argument. We have a local existence result.

Proposition 4.2 *Assume $\phi_0 \in H_Q^\infty$. Then there is $T_{\sigma_0} = T(\|\phi_0\|_{H_Q^{\sigma_0}}) > 0$ and a solution $\phi \in C([-T_{\sigma_0}, T_{\sigma_0}]; H_Q^\infty)$ of the initial value problem (1.1). Additionally, T_{σ_0} can be chosen so that*

$$\sup_{t \in [-T_{\sigma_0}, T_{\sigma_0}]} \|\phi(t)\|_{H_Q^{\sigma_0}} \leq C(\|\phi_0\|_{H_Q^{\sigma_0}}), \tag{4.22}$$

and for $\sigma \in [\sigma_0, \infty) \cap \mathbf{Z}$,

$$\sup_{t \in [-T_{\sigma_0}, T_{\sigma_0}]} \|\phi(t)\|_{H_Q^\sigma} \leq C(\sigma, \|\phi_0\|_{H_Q^\sigma}). \tag{4.23}$$

Proof: See [9]. \square

Then for $\phi_0 \in H_Q^{\sigma_0}$, for σ_0 sufficiently large, and $T(\|\phi_0\|_{H_Q^{\sigma_0}}) > 0$ sufficiently small, we have that

$$\sum_k \gamma_k \lesssim \epsilon, \tag{4.24}$$

for $\epsilon(\|\phi_0\|_{\dot{H}^{d/2}})$ sufficiently small. Then by theorem 3.1 and proposition 4.1,

$$\sum_k \alpha_k^2 \lesssim \|\phi_0\|_{\dot{H}^{d/2}}^2, \quad (4.25)$$

and

$$\sup_k \alpha_k + \sup_k \gamma_k \lesssim \epsilon. \quad (4.26)$$

Again by theorem 3.1 and proposition 4.1,

$$\alpha_j \lesssim \alpha_j(0) + \gamma_j, \quad (4.27)$$

and

$$\gamma_j \lesssim \epsilon \alpha_j + \epsilon \gamma_j. \quad (4.28)$$

Combining (4.27) and (4.28),

$$\alpha_j \lesssim \alpha_j(0), \quad \text{and} \quad \gamma_j \lesssim \epsilon \alpha_j. \quad (4.29)$$

But we could then replace α_j and γ_j by $\tilde{\alpha}_j$ and $\tilde{\gamma}_j$, with

$$\tilde{\alpha}_j = \sup_k 2^{-\delta|j-k|} 2^{k(\sigma_0-1)} \|P_k \psi_x\|_{L_t^\infty L_x^2}, \quad (4.30)$$

and

$$\tilde{\gamma}_j = \sup_k 2^{-\delta|j-k|} 2^{k(\sigma_0-1)} \sup_l (2^{k-l} \|(P_k \psi_x)(P_l \psi_x)\|_{L_{t,x}^2}) \quad (4.31)$$

Following the arguments proving theorem 3.1 and proposition 4.1, it is possible to prove that

$$\tilde{\alpha}_j \lesssim \tilde{\alpha}_j(0) + \epsilon \tilde{\gamma}_j, \quad (4.32)$$

and

$$\tilde{\gamma}_j \lesssim \epsilon \tilde{\alpha}_j + \epsilon \tilde{\gamma}_j, \quad (4.33)$$

and therefore

$$\tilde{\alpha}_j \lesssim \tilde{\alpha}_j(0). \quad (4.34)$$

In particular, this implies that for any $t \in [-T_{\sigma_0}, T_{\sigma_0}]$,

$$\|\phi(t)\|_{\dot{H}^{\sigma_0}} \lesssim \|\phi_0\|_{\dot{H}^{\sigma_0}}. \quad (4.35)$$

Then we may extend the interval of existence a little farther, but still maintaining (4.24), which then implies that (4.35) holds. We can iterate this argument, and the implicit constants in (4.29) and (4.35) are uniformly bounded. Making a standard bootstrap argument then implies theorem 1.3. \square

References

- [1] I. Bejenaru, “Global results for Schrödinger maps in dimensions $n \geq 3$ ”, *Communications in Partial Differential Equations* **33** (2008) pp. 451 – 477.
- [2] I. Bejenaru, A.D. Ionescu, and C. E. Kenig, “Global existence and uniqueness of Schrödinger maps in dimensions $d \geq 4$ ”, *Advances in Mathematics* **215** (2007) pp. 263 – 291.
- [3] I. Bejenaru, A.D. Ionescu, C. E. Kenig, and D. Tataru, “Global Schrödinger maps in dimensions $d \geq 2$: small data in the critical Sobolev spaces”, *Annals of Mathematics* **173** (2) no. 3 (2011) pp. 1443 – 1506.
- [4] J. Bourgain, “Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity”, *International Mathematical Research Notices*, **5** (1998) pp. 253 – 283.
- [5] J. Colliander, M. Grillakis, and N. Tzirakis. “Tensor products and correlation estimates with applications to nonlinear Schrödinger equations”, *Communications on Pure and Applied Mathematics*, **62** no. 7 (2009) pp. 920 – 968.
- [6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, “Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on \mathbf{R}^3 ”, *Communications on Pure and Applied Mathematics*, **21** (2004) pp. 987 - 1014.
- [7] B. Dodson, “Bilinear Strichartz estimates for the Schrödinger map problem”, Arxiv eprints: 1210 : 5255 (2012).
- [8] B. Dodson and P. Smith, “A controlling norm for energy - critical Schrödinger maps”, *Transactions of the American Mathematical Society* **367** (2015) pp. 7193 – 7220.
- [9] C. Kenig, T. Lamm, D. Pollack, G. Staffilani, and T. Toro, “The Cauchy problem for Schrödinger flows into Kähler manifolds”, Preprint.
- [10] J. Krieger, “Global regularity of wave maps from \mathbf{R}^{3+1} to surfaces”, *Communications in Mathematical Physics* **238** (2003) pp. 333 – 366.
- [11] J. Krieger, “Global regularity of wave maps from \mathbf{R}^{2+1} to \mathbf{H}^2 . Small energy”, *Communications in Mathematical Physics* **250** (2004) pp. 507 – 580.
- [12] A. Nahmod, A. Stefanov, and K. Uhlenbeck, “On the well - posedness of the wave map problem in high dimensions”, *Comm. Anal. Geom.* **11** (2003) pp. 49 – 83.

- [13] F. Planchon and L. Vega, “Bilinear virial identities and applications” *Annales Scientifiques de l’École Normale Supérieure Quatrième Série* **42** no. 2 (2009) pp. 261 - 290.
- [14] J. Shatah and M. Struwe, “The Cauchy problem for wave maps”, *International Mathematics Research Notices* **2002** (2002) pp. 555 – 571.
- [15] P. Smith, “Global regularity of critical Schrödinger maps : subthreshold dispersed energy”, Arxiv eprints : 1112.0251 (2011).
- [16] P. Smith, “Geometric renormalization below the ground state”, *International Mathematics Research Notices* **16** (2012) pp. 3800 – 3844.
- [17] P. Smith, “Conditional global regularity of Schrödinger maps : subthreshold dispersed energy”, *Analysis of PDE* **6** (2013) pp. 601 – 686.
- [18] T. Tao “Global regularity of wave maps II. Small energy in two dimensions”, *Communications in Mathematical Physics* **224** (2001), pp. 443 – 544.
- [19] T. Tao, M. Visan, and X. Zhang. “The nonlinear Schrödinger equation with combined power-type nonlinearities”, *Communications in Partial Differential Equations*, **32** no. 7-9 (2007) pp. 1281–1343.